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BOOK REVIEW

Vorticity and Incompressible Flow. By A. J. Majda & A. L. Bertozzi. Cambridge University Press, 2001. 558 pp. ISBN 0521630576, £75.00 (hardback); ISBN 0521639484, £27.95 (paperback).

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This book is destined to become a classic. A serious graduate student who wants to learn the modern, highly technical methods of mathematical analysis that have been pioneered in the last few years to investigate the subtle dynamics of vortical structures in inviscid and viscous turbulence will find all of the necessary tools in this book. Predominantly it is a book about the two-dimensional and three-dimensional incompressible Euler equations and their associated weak solution formulations, although a substantial amount of Navier–Stokes analysis exists between its covers.

The first chapter begins with elementary properties of vorticity and some of the simpler exact solutions of the Euler and Navier–Stokes equations that can be gleaned from studying special forms of the velocity gradient and strain matrices, as well as considering the effects of vortex stretching, convection and diffusion. Conserved quantities are also discussed. It is at this point early in the book that the reader is introduced to Leray's formulation of the Navier–Stokes equations using the idea of projecting onto divergence-free vector fields (Leray 1934).

The second chapter studies the vorticity–stream formulation of the Euler and Navier–Stokes equations and considers some exact, steady solutions of two-dimensional Euler equations, such as some periodic flows and Kelvin–Stuart cat's eyes (Stuart 1971). Among the three-dimensional flows with non-trivial dynamics that are exhibited are a restricted class of Burgers-like solutions (an example of what the authors call two and a half dimensional flows), ABC flows and axisymmetric flows. Given the size of the book (545 pages) it was probably a wise decision on the part of the authors to limit the extent of the examples and not make an extensive catalogue of exact solutions of the two-dimensional and three-dimensional Euler and Navier–Stokes equations. The student can find such material in more conventional literature. A substantial section at the end of the chapter is devoted to particle trajectory methods.

In Chapter 3 the student is introduced to basic energy estimates that establish existence of solutions local-in-time. Methods are introduced here that are used extensively later: inequalities in norms, calculus and Gagliardo–Nirenberg inequalities, Gronwall's Lemma and the idea of mollification by regularization of equations to produce smooth solutions. The student is pointed to direct textbook references where these standard methods of analysis are explained.

Chapter 4 makes use of results from Chapter 2 on particle trajectory methods. It is shown that standard existence and uniqueness theories for such equations yield local-in-time existence and uniqueness of solutions of the Euler equations. Global-in-time existence for both three-dimensional Euler and the Navier–Stokes equations, established in Chapter 3, had been based on a sufficient condition on the accumulation of vorticity; this chapter revisits these results using Hölder norms and potential theory. From this can be found global-in-time existence for smooth solutions of the two-dimensional Euler equations and for axisymmetric without swirl solutions of the three-dimensional Euler equations.

Chapter 5 is devoted to one of the great issues of modern applied mathematics: Do the three-dimensional Euler equations develop a finite time singularity from smooth initial data? Vorticity clearly accumulates in a very strong fashion but whether it actually blows up in a finite time is still an open question. The pre-eminent theoretical result in this area is the celebrated Beale–Kato–Majda (BKM) theorem (Beale, Kato & Majda 1984). One particular version of the theorem says that if $\|\omega\|_\infty$ is the peak vorticity and if $\int_0^{t^*} \|\omega\|_\infty \,\mathrm{d}t$ is finite at some time t^*, then no three-dimensional Euler variable of any sort (e.g. arbitrarily large derivatives of the velocity field or even $\|\omega\|_\infty$ itself) can be singular at t^*. The theorem itself does not predict a singularity; to date no proof exists that shows that the three-dimensional Euler equations become singular or remain regular. The theorem nevertheless has some powerful consequences: suppose that a reader, in a heady moment, becomes bent on integrating the three-dimensional Euler equations by some numerical scheme and finds that the peak vorticity blows up like $\|\omega\|_\infty \sim (t^* - t)^{-\alpha}$. To be consistent with the BKM-theorem, the exponent must obey $\alpha \geqslant 1$ but if it lies in the range $0 < \alpha < 1$ then the supposed singularity must be an artefact of the chosen numerical scheme. In fact several very early attempts did indeed produce values of α in the range $0 < \alpha < 1$. The theorem is therefore a guide to the computational fluid dynamicist as it says that the only variable that needs to be monitored is $\int_0^{t} \|\omega\|_\infty \,\mathrm{d}t$. The first large-scale Euler calculation with singular behaviour that was consistent with the BKM theorem was performed by Robert Kerr (Kerr 1993), using anti-parallel vortex tubes as initial data, who found $\alpha = 1$. Since then Constantin, Fefferman & Majda (1996) have looked at the direction of three-dimensional Euler vorticity as well as its magnitude, by considering the relative orientation of neighbouring vortex lines. Other attempts have been made to understand vorticity accumulation processes through analytic analogies to three-dimensional Euler equations: the two-dimensional quasi-geostrophic model of Constantin, Majda & Tabak (1994) is one of these in which the level sets of a scalar θ are analagous to vortex lines of the three-dimensional Euler equations.

Chapter 6 is a compendium of numerical methods: the random vortex method for viscous strained shear layers, two-dimensional/three-dimensional inviscid-vortex methods and their convergence properties, the performance of the latter method in two dimensions is assessed on a model problem and finally there is a section on the two-dimensional random vortex method.

Over the years, modelling of the behaviour of vortex filaments has been a popular and enduring subject. The reviewer lives under the Heathrow Airport landing flightpath and regularly observes the Crow instability of the trailing vortices from aircraft wings. The behaviour of vortex filaments constitutes Chapter 7 of the book. Under the self-induction approximation, the Hasimoto transformation turns the equations for a vortex filament into the nonlinear Schrödinger equation, which is exactly solvable by the inverse scattering method. The authors correctly point out that the self-induction approximation precludes any self-stretching so if other effects are to be included, such as bending and folding of a filament, including local self-stretching, then other asymptotic models are needed with new assumptions. The rest of the chapter discusses various circumstances such as the properties and stability of the filament equation when there is a background flow. The occurrence of kinks and hairpins in filaments, including some of Chorin's early work (Chorin 1982), is also discussed, as are the dynamics of nearly parallel filaments. Interestingly – and helpfully for the serious graduate student – there is a section at the end of the chapter (§7.5) where the authors lay out a set of interesting open problems in this area.

As the reader moves into the material in Chapters 8–12 it will become noticeable that the book changes gear and moves to a more advanced level. The material deals with issues related to non-smooth solutions of the Euler equations appropriate to modelling highly unstable vortical structures. Chapter 8 deals with two-dimensional vortex patches in which the vorticity is considered to be a constant $\omega = \omega_0$ within a bounded time-evolving domain Ω but $\omega = 0$ outside Ω. Hence the vorticity is discontinuous and represents eddy-like motion in one region, with the flow being essentially irrotational elsewhere. To study these patches of constant vorticity requires the reformulation of the two-dimensional Euler equations in a weaker form. Chemin (1993), subsequently followed by a simpler proof by Bertozzi & Constantin (1993), settled the longstanding issue of whether the boundary of Ω can develop a cusp in finite time by showing that initially smooth boundaries must remain smooth.

Chapter 9 introduces the idea of vortex sheets (which have a velocity discontinuity) relevant to jets, wakes and mixing layers. This requires the idea of even weaker solutions of the Euler equations. Whereas a vortex patch has bounded vorticity pointwise within the patch, the vorticity in a sheet is a δ-function along a surface of co-dimension unity. This is a curve in two dimensions and a surface or sheet in three dimensions. Most of the chapter concentrates on the two-dimensional case. The material in the chapter covers the classical Birkhoff–Rott equation, the Kelvin–Helmholz instability and a survey of computational methods such as the point vortex method and the various regularizations that have been employed.

The ill-posedness of vortex sheets and the issue of how the complex dynamics on them is smoothed at fine scales by residual viscosity is studied in detail in Chapters 10–12. The study of approximate-solution sequences for the Euler equations is developed in these chapters. Chapter 11 takes this idea and studies the problem of the two-dimensional Euler equations with vortex sheet initial data. In § 11.4 it is shown that there is a solution of the two-dimensional Euler equations when the vorticity has a distinguished sign; the material here is based on the proof by Delort (1991) and Majda (1993).

Chapter 12 develops the idea of approximate-solution sequences and how they can be used to study the vorticity concentration of fine-scale structures in the zero viscosity limit. Here the vorticity is in three dimensions and has mixed sign. A way is devised of measuring the set (in the Hausdorff sense) on which the concentration takes place. Young measures are introduced to preserve the idea of a solution in a very weak sense even when oscillations take place. The material in Chapters 11 and 12 is based on the joint work of DiPerna and Majda from the late 1980s. Chapter 13 spreads the scope of the material in the book further by briefly considering the Vlasov–Poisson equations for collisionless plasmas. There are analogies between vortex sheets and electron sheets and vortex patches and electron patches.

Euler (1711–1783) himself, if present today, might be amazed at the complexity of the dynamics displayed by solutions of the innocent-looking equations he wrote down. Majda and Bertozzi have produced a formidable and extremely well-written book on this subject that throws out a series of challenges to the modern student. They have demonstrated an inventive mastery over a wide class of advanced mathematical material and methods that the student must also master if he/she is to get to grips with the highly complex phenomena that occur at fine scales in fluid dynamics. The chapter notes point the student to basic textbook material relevant to the understanding of the chapter in question. The preface indicates that the book grew from extensive graduate lecture material given by Andrew Majda at Princeton University and the Courant Institute over a period of years. This is the standard to which the rest of us need to aspire.

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SHORT NOTICES

Tubes, Sheets and Singularities in Fluid Dynamics. Edited by K. Bajer & H. K. Moffatt. Kluwer, 2002. 379 pp. ISBN 1402009801. £72.00.

Originating from a Symposium held in Zakopane, Poland in September 2001, this book contains forty-three papers by various authors and provides a comprehensive overview of current research into vortex dynamics. The individual contributions are rather short (typically six pages) but many of them are by leading figures in the area. The book is divided into six parts:

(i) vortex structure, stability and evolution;
(ii) singular vortex filaments;
(iii) magnetic structure, topology and reconnection;
(iv) vortex structures in turbulent flows;
(v) finite-time singularities;
(vi) Stokes flow and singular behaviour near boundaries.

Attractively produced by Kluwer, this book and will be of particular interest to those who are active in vortex dynamics research.

Ferrofluids: Magnetically Controlled Fluids and Their Applications. Edited by S. Odenbach. Springer Lecture Notes in Physics, 2002. 251pp. ISBN 3540439781. £47.

Ferrofluids are suspensions of magnetic nanoparticles and have been around since the early sixties. They exhibit magnetic properties and so can be controlled by magnetic fields. This book is based on a series of plenary talks given at a conference held in Bremen in 2001 and contains twelve contributions by different authors. The articles develop four main themes: the synthesis of ferrofluids; microscopic and continuum theories; rheological properties such as magnetoviscosity; and medical applications. The book is intended to be both an introduction to, and state-of-the-art review of, ferrofluids.

Large-Scale Atmosphere-Ocean Dynamics, Volume 1: Analytical methods and numerical models. Edited by J. Norbury & I. Roulstone. Cambridge University Press, 2002. 370pp. ISBN 052180681. £50.

This book has its origins in two meetings held at the Isaac Newton Institute in Cambridge in 1996 and 1997, both devoted to the dynamics of the atmosphere and the oceans. It is the first of two volumes, and sets out to provide a contemporary account of the mathematics and numerical modelling of weather forecasting, climate change, dynamic meteorology and oceanography. The book is aimed at mathematicians and meteorologists and contains six articles by different authors. The contributions vary in length from twenty-five pages to one hundred pages. There is no index.

Vascular Grafts: Experiment and Modelling. Edited by A Tura. WIT Press, Advances in Fluid Mechanics, 2003. 421pp. ISBN 1853129003. £138.00.

Cardiovascular disease associated with abnormal blood flow in arteries represents one of the major causes of death in developed countries. This book sets out to provide an extensive summary of haemodynamic and biomechanical aspects of graft replacement and their influence on the success or failure of a graft implantation. It covers both experimental studies and mathematical modelling and is aimed at researchers in biomechanics. The book comprises ten research articles by various authors, many of whom have an engineering or bioengineering background.